Groundings Undergraduate Academic Journal

University of Glasgow | Glasgow University Union

Modern Solar Navigation Techniques

Author(s): Alberto Bodas Gallego

Source: Groundings Undergraduate, April 2023, Vol. 14, pp. 29-50

Published by: University of Glasgow, Glasgow University Union Publications

ISSNs: 1754-7474 (Print) | 1755-2702 (Online)

Modern Solar Navigation Techniques

Alberto Bodas Gallego

Navigation by sea has proven difficult due to the absence of distinct markers for guidance. One solution for longer journeys was to track the position of celestial bodies as a navigational method, which has become more reliable as mathematical models improved over time. This essay aims to explore the mathematical methods behind modern solar navigation techniques and illustrate how these models are highly useful to describe and predict real-life scenarios.

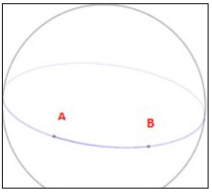
Introduction

Navigation is defined as "the science of getting ships, aircraft or spacecraft from place to place." It is especially concerned with determining the position and course of a vessel, to safely plan, track and control the craft's journey.
Navigating by sea has always been especially complicated, due to the absence of distinctive landmarks to guide your course. Once sailors began undertaking longer journeys, losing sight of land, they required a system of navigation that would be viable in open sea. The solution was to measure the position of celestial bodies to determine the observer's location using appropriate mathematical tools.[4]

Using the sun and stars for this purpose is an elegant and ingenious solution, and understanding the motion of celestial bodies has been the driving force in physical development through some of the most significant scientific expansions in history. Furthermore, celestial navigation constitutes important knowledge on a light vessel, as it is always reliable in case of electronic equipment failure.
Hence, this essay will explore some mathematical methods used to navigate the globe using the sun and develop some alternative solar methods which can be used for navigation.

Navigation

Some modelling assumptions are made in this essay which must be explained. These are grouped according to their effect.
This paper assumes that the earth is a perfect sphere (as opposed to an ellipsoid [10]), and that the earth is stationary during a day (as opposed to rotating 0.985° around the sun).



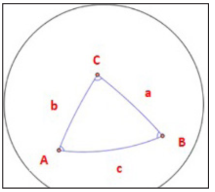


Figure One (Left): Great Circle (and segment) through A and B. **Figure Two (Right):** Spherical Triangle drawn by segments a, b and c.

Both these effects are negligible [10]. Furthermore, Polar Precession, which is the rotation of the Rotational Axis about the Orbital Axis once every 2,600 years [9], is ignored as it has negligible effects during our lifetimes.

Finally, this paper assumes the light rays incident on the earth from the sun are parallel. The average distance from the earth to the sun, defined as one AU, is 150,000,000km. In comparison, the radius of the earth is only 6,371km on average [1]. This is a ratio of 1: 23 544, which means the radius of the earth can be ignored, justifying this assumption.

More significant assumptions include ignoring the earth's ellipsoidal orbit, which noticeably affects the angular speed of the earth around the sun, and therefore declination calculations, at some times of the year (see [8]). Atmospheric refraction (the bending of light as it enters the atmosphere) is also ignored. Refraction causes an average of 1.2° per day of additional sunlight [1] (which is roughly equivalent to a few additional minutes), with variable effect. Finally, magnetic north is assumed

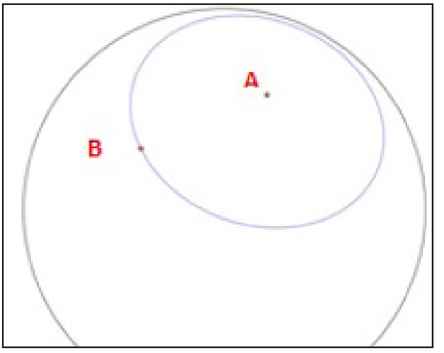


Figure Three: Small circle of centre A, passing through B.

to lie exactly above the north pole, which is not necessarily the case [2]. This effect, called Magnetic Deviation, is location-dependent and there exist local corrections, but it is ignored in this essay.

Terminology

Spherical Geometry

A Great Circle is the circle drawn on the surface of a sphere by a plane intersecting that sphere, which passes through the sphere's centre, such as the equator.

A Segment is the shortest line connecting two points on a sphere's surface, which is always a finite section of a great circle. Any two points on a sphere can be connected by exactly one great circle segment unless they are antipodes. As all spheres are similar, segment lengths are simplified to the angles subtending them from the sphere's centre. Angular lengths are used for all spherical trigonometry formulae and can be converted to real lengths using the radius of the sphere in question.

A Spherical Triangle is a triangle drawn on the surface of a sphere by three intersecting segments. The lengths of its sides and the angles between them can be resolved using spherical rules analogous to the planar sine and cosine rules.

A Small Circle is the circle drawn on the surface of a sphere by a plane intersecting the sphere without passing through the centre.
All parallels (except the Equator) are small circles, with the Rotational Axis passing through their centres.
An Antipode is the diametrical opposite of a given point on a sphere, which is a pair of points through which a straight line can be passed through which also goes through the origin.

Navigation

Any location on a sphere's surface, in this case the Earth, can be described by two angles, measured from the sphere's centre, O, perpendicular to each other. These angles are analogous to the x and y coordinates on a Cartesian plane.

On Fig.4, the position of vessel V is defined by angle λ, called latitude, and angle, φ called longitude.

$$V = (\lambda, \varphi)$$

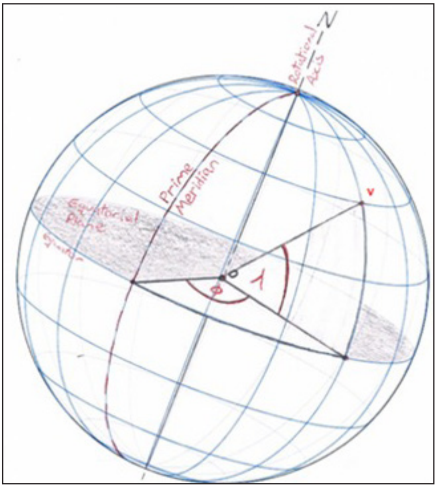


Figure Four : Small circle of centre A, passing through B.

Latitude, λ: The angle measuring the North-South displacement of a given point, defining the Equator as λ= 0°. North is defined as the positive direction and South as the negative direction. Therefore, latitude oscillates from -90° to +90° at the poles. The small circles normal to the Rotational Axis (NS) which mark different latitudes on a map are called Parallels.

Longitude, φ : The angle measuring the East-West displacement of a given point, using the Prime Meridian (passing through Greenwich, England) as λ= 0° . West of the Prime Meridian is defined as negative and East of the Prime Meridian as positive. Therefore, Longitude ranges from -180° to +180°. (Note that -180° and +180° longitude are equivalent, both marking points on the 180° meridian.) The semicircles running from the North to South poles, which mark different longitudes on a map, are called Meridians.

Note that whereas the Equator is a natural choice for 0° λdue to Earth's rotation, the choice of the Prime Meridian for 0° λis completely arbitrary.

This paper adopts mathematical convention, where positive and negative signs are used to differentiate north from south and east from west (opposed to the use of letters N, E, S and W (33°N, 32°W) seen in ocean charts), and decimals are used when being accurate to more than one degree, opposed to minutes and seconds.

Note also that the symbols for latitude and longitude are switched around in charts [11]. To convert between decimals and seconds and minutes, use the definitions

$$1' = \left(\frac{1}{60}\right)^{\circ} \text{ and } 1'' = \left(\frac{1}{360}\right)^{\circ}$$

(chosen so there are 60 minutes in a degree and 60 seconds in a minute.) A nautical mile is defined as the distance on the earth's surface in a nautical minute, so 1NM=1.852km. [17]

Horizon Plane, h: A plane tangential to the sphere's surface at point V. As light travels in straight lines, any celestial body

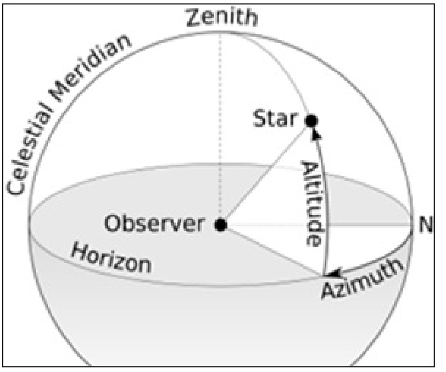


Figure Five : Altitude and Azimuth

above this plane is visible to an observer at V, and any point below it is not.

Celestial Sphere: an imaginary sphere with used to map the position of celestial bodies. Two systems of coordinates exist: *Equatorial coordinates*: earthly coordinates (latitude and longitude) are projected onto the celestial sphere, and the centre of the Celestial Sphere is O, the centre of the Earth. [14]

Horizontal coordinates: coordinates relative to the observer's horizon plane are used, with the observer in the centre of the sphere. [7]

Azimuth, α:Measures the horizontal angular displacement of a body from the observer's north bearing.
Altitude, e: Measures the vertical angular displacement of a body.

Zenith: The point on the celestial sphere at 90° Altitude.

Note that (based on the Celestial Parallel Assumption) it is assumed V and O are on the same point, to allow the two coordinate systems to be compared.

Calculating Longitude using Time

Whereas finding latitude through the altitude of stars or the sun was widely used since ancient times, calculating longitude accurately was not possible until the chronometer was invented. [4]
The earth completes one full rotation around its axis in 23 hours and 56 minutes, such that it has the same orientation relative to the rest of the universe. This is called a Sidereal day, used when dealing with stellar motion. [19]
However, as the earth is rotating around the sun, it takes another 4 minutes for the sun to reach the same meridian it started the day at, compensating for the 0.985° travelled around the sun in this time. Accounting for both these effects, it takes 24 hours for the earth to rotate 360° around the sun in what is known as a Solar day, used when dealing with solar motion. [19]

$$\frac{360°}{24h} = 15°/h$$

The Meridians every 15° mark different time zones. This simple calculation also means that if the solar time of a reference point (GMT, Greenwich Mean Time, is used) is known through an accurate chronometer, and Local Solar Time (LST) at the observer's longitude is calculated, the difference in time can be converted to degrees, obtaining a result for longitude.

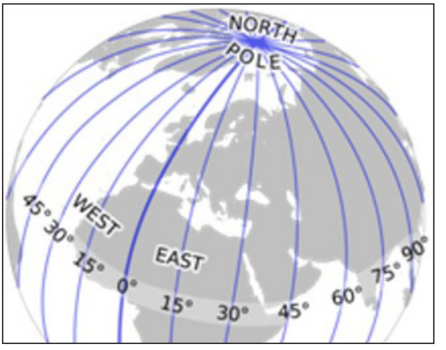


Figure Seven : World Meridians

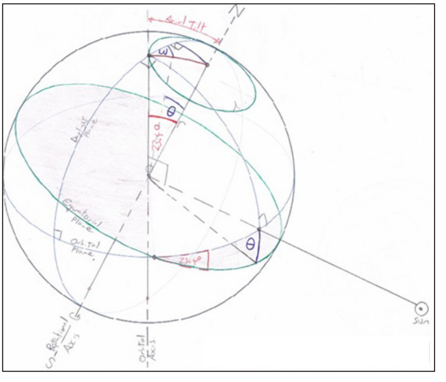


Figure Eight : Axial Tilt, the Daylight Plane and the Declination

$$\varphi = 15 * (GMT - LST)$$

LST was typically measured at Solar Noon, because when

$$\alpha(sun) = 0° \text{ or } 180°, LST = 12.00$$

by convention, making it the most convenient time for measurement.

Calculating Solar Declination (Θ)

Being able to calculate Solar Declination is vital knowledge for all modern location calculations. Declination depends on Axial Tilt and the position of the Daylight Plane.
Axial Tilt: The angle between the Equatorial Plane and the Orbital Plane. (Also the angle between the Rotational Axis (NS) and the Orbital Axis)

Equatorial Plane: The plane projected from $\lambda=0°$.

Orbital Plane: The plane the Earth orbits the Sun through.

As the earth orbits, the North and South poles' orientation remains fixed with respect to the rest of the universe but rotates with respect to the sun. Whether the North or South Pole is facing the sun is responsible for Earth's seasons. [13]

$$Axial\ tilt = 23.4°$$

The Daylight Plane is the plane normal to the line ⨀, which joins O, the centre of the earth and ⨀, the centre of the sun. It divides the earth into two hemispheres, one which is facing the sun and has daylight, and one facing away from the sun, which has night-time. This plane is independent of the earth's rotation, meaning that whilst points on earth rotate at 15°/h around NS, ⨀O and the Daylight Plane remain fixed relative to the sun. Solar Declination (Θ) is the angle between NS and the Daylight Plane,

Planet Earth : Great Circle marked by the Daylight Plane

which is also the latitude over which the sun reaches local Zenith at midday.
NS can be thought of as rotating around the Orbital Axis, completing 1 revolution a year. This means that the distance from the rotational axis at N to the daylight plane can be described as:

$$R = r * \sin(23.4)$$

$$l = R * \sin(\omega)$$

$$l = r * \sin(23.4) * \sin(\omega)$$

Where l=the distance from N to the Daylight Plane, R=the distance from N to the Orbital Axis and ω= the angle of displacement of the earth around the sun, measured from the last spring equinox The spring equinox used for measurements in this paper took place on the 21/03/2019 [20].

The spring equinox was a natural choice for ω=0, because at that date, NS is aligned with the Daylight Plane, meaning Θ=0 when ω=0.

$$\theta = \arcsin\left(\frac{l}{r}\right)$$

$$\boldsymbol{\theta = \arcsin(\sin(23.4) * \sin(\omega))}$$

It should be noted that these constructions show Θ ranges between −23. 4° and 23. 4°. ω can be converted to days by introducing the conversion factor

$$\omega = \frac{360}{365.25}\Omega:$$

$$\boldsymbol{\theta = \arcsin\left(\sin(23.4) * \sin\left(\frac{360}{365.25}\Omega\right)\right)}$$

Where Ω= days since the last spring equinox.

Calculating Latitude using Solar Height at Noon

One of the most widespread methods of calculating latitude is measuring the sun's altitude at solar noon, when the sun is aligned with the observer's meridian and the centre of the earth [6]. The sun reaches its maximum altitude for the observer at this time. ⨀V and ⨀O are assumed to be parallel.

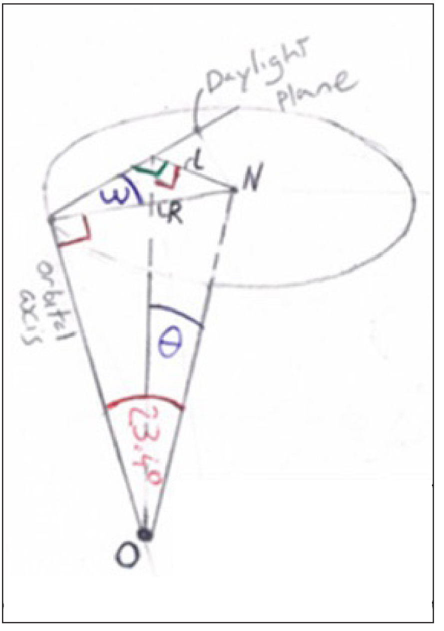


Figure Ten : Constructions for calculating Θ

$$a = 90 - e$$

$$\lambda = a + \theta$$

$$\lambda = 90 - e + \theta$$

However, for a λ in the Southern Hemisphere, λ ≠ a + Θ. Instead:

$$\lambda = a - \theta$$

This ambiguity limits this method, as the observer may not know which Hemisphere they are in. Measuring the elevation of the sun, instead of from the closest azimuth, only from α=180° (South), can resolve this issue:

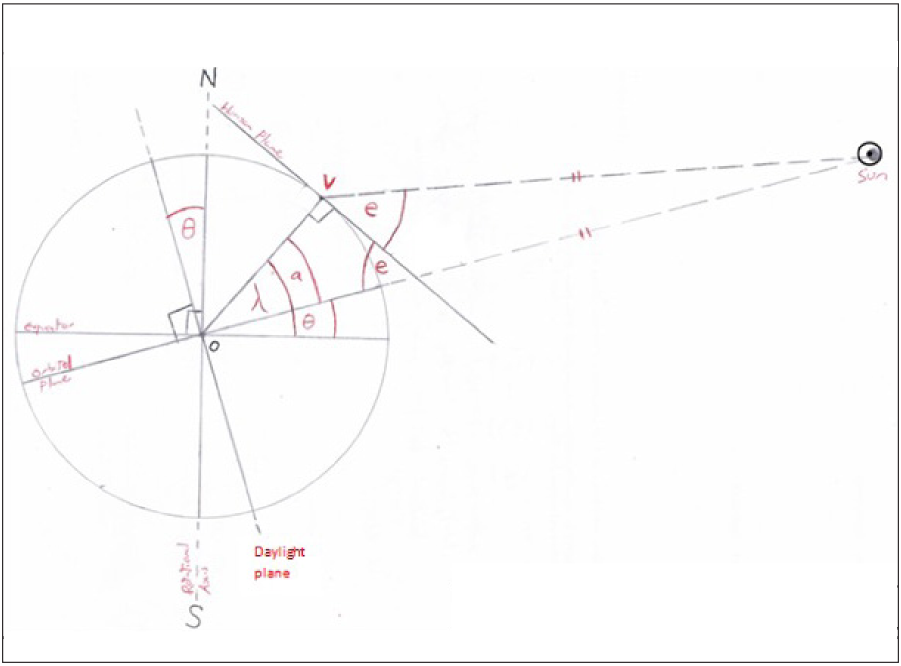


Figure Eleven : A cross section of the Earth at Solar Noon

$$e_S = solar\ elevation\ from\ south$$

Where: $e_S = 180 - e_N$

Leading to: $\boldsymbol{\lambda = 90 - e_S + \theta}$

This variation of the formula can be used from any latitude without ambiguity.

Calculating Latitude through the Daylight Angle

With modern knowledge about the solar system, navigators can find their location in more innovative ways. One such way is through measuring the Daylight Angle.

The two intersections between parallel λV and the daylight plane mark the points of sunrise and sunset for V, which could be any point on this latitude. This means that angle Ψ is the angle of rotation around NS for which observers at λV are in the dark, and its pair, Ψ, is the angle for which observers have daylight:

$$\Psi = 15 * (GMT_{sumset} - GMT_{sunrise})$$

$$\psi = 15 * (GMT_{sumrise} - GMT_{sunset})$$

These angles are dependent on Θ and λ, with effects that can be qualitatively summarised. When Θ is large, the difference between Ψ and   increases,

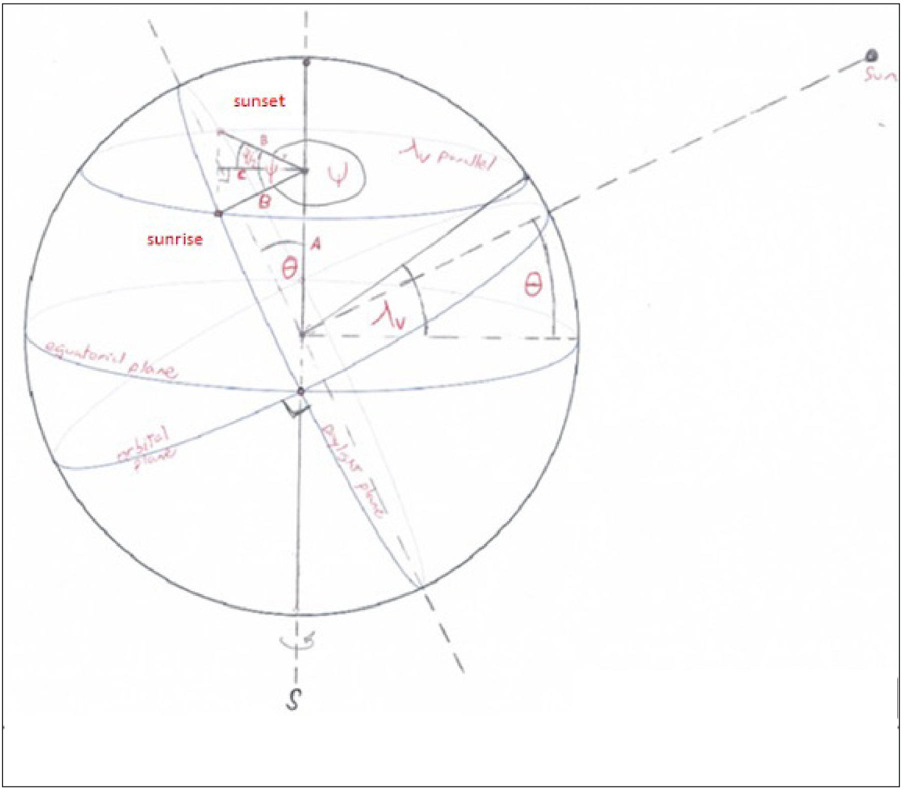


Figure Twelve : The Daylight Angle

and when λ is large, the difference between Ψ and increases. When either λ or Θ (or both) tend to zero, Ψ and both tend to 180°. (Hence, points on the Equator have exactly 180° (12 hours) of sunlight every day of the year.)
These observations explain some properties of the earth.
When Θ=0°, every latitude has exactly 180° (12 hours) of sunlight, which happens two days a year, on the Equinoxes. There are also two solstices every year, one being the longest and the other the shortest day of the year. They occur when Θ is largest (Θ=±23.4°). As can be appreciated from Fig.12, there are very large positive and negative latitudes where the Daylight Plane does not intersect parallel λ. Points with latitude greater than these would have no sunrise or sunset, therefore having 360° of full daylight or night-time for at least a day a year. The Polar Circles, which can be found at 66°33′47.8″ N and 66°33′47.8″ S [12], are the closest parallels to the equator where this occurs.

Using these constructions, an expression for can be constructed from these two angles. Firstly, lengths A, B and C are defined to construct triangles involving Θ and λ.

$$A = r * \cos(90 - \lambda)$$

Which can be rewritten as:

$$\mathbf{A} = \boldsymbol{r} * \mathbf{sin}\ (\boldsymbol{\lambda})$$

(Recall A=the distance from O to the intersection of the rotational axis with λ)

$$B = r * \sin(90 - \lambda)$$

Similarly, B can be simplified to

$$\boldsymbol{B} = \boldsymbol{r} * \mathbf{cos}\ (\boldsymbol{\lambda})$$

(B=the radius of the small circle defined by the parallel at latitude λ)

$$\boldsymbol{C} = \boldsymbol{A} * \mathbf{tan}\ (\boldsymbol{\theta})$$

(C=the bisector of angle ψ, which joins the intersection between NS and λ with the intersection between the Orbital Axis and λ)

From the right-angled triangle formed with B, C and $\Psi/2$, we can conclude that:

$$\psi = 2 \arccos\left(\frac{C}{B}\right)$$

Substituting for our lengths and simplifying, we get

$$\psi = 2 \arccos\left(\frac{A * \tan(\theta)}{r * \cos(\lambda)}\right)$$

$$\psi = 2 \arccos\left(\frac{r * \sin(\lambda) * \tan(\theta)}{r * \cos(\lambda)}\right)$$

$$\boldsymbol{\psi} = \mathbf{2\ arccos}(\mathbf{tan}(\boldsymbol{\theta}) * \mathbf{tan}\ (\boldsymbol{\lambda}))$$

We can rearrange for λ, to obtain

$$\boldsymbol{\lambda} = \mathbf{arctan}\left(\frac{\mathbf{cos}\left(\frac{\boldsymbol{\psi}}{\mathbf{2}}\right)}{\mathbf{tan}(\boldsymbol{\theta})}\right)$$

As Ψ = 360° −ψ, we can substitute for ψ to obtain the daytime equivalent

$$\lambda = \arctan\left(\frac{\cos\left(180 - \frac{\Psi}{2}\right)}{\tan(\theta)}\right)$$

$$\boldsymbol{\lambda} = \mathbf{arctan}\left(-\frac{\mathbf{cos}\left(\frac{\boldsymbol{\Psi}}{\mathbf{2}}\right)}{\mathbf{tan}(\boldsymbol{\theta})}\right)$$

Rearranging for Ψ

$$\boldsymbol{\Psi} = \mathbf{2arccos}\ (-\,\mathbf{tan}(\boldsymbol{\lambda}) * \mathbf{tan}(\boldsymbol{\theta}))$$

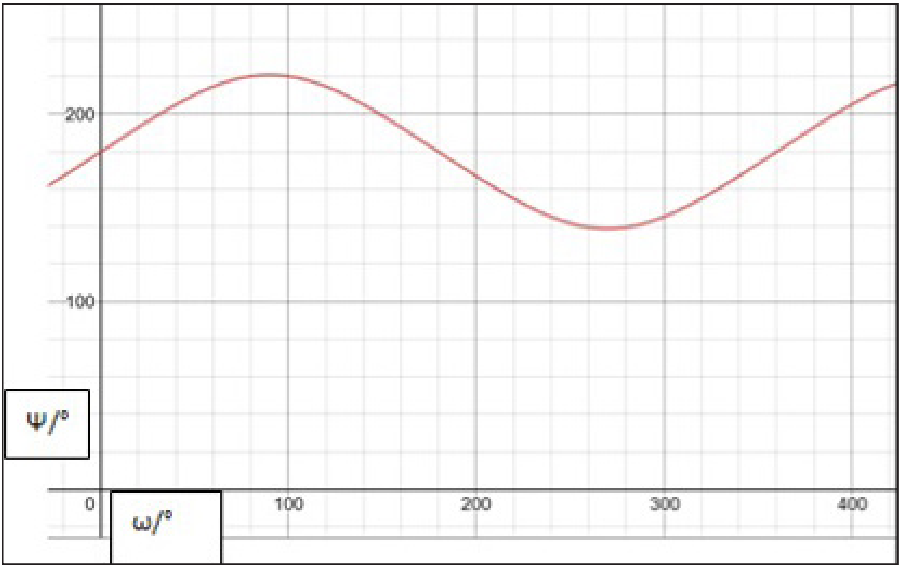


Figure Thirteen : The Yearly Angle against the Daylight Angle. λ=39°

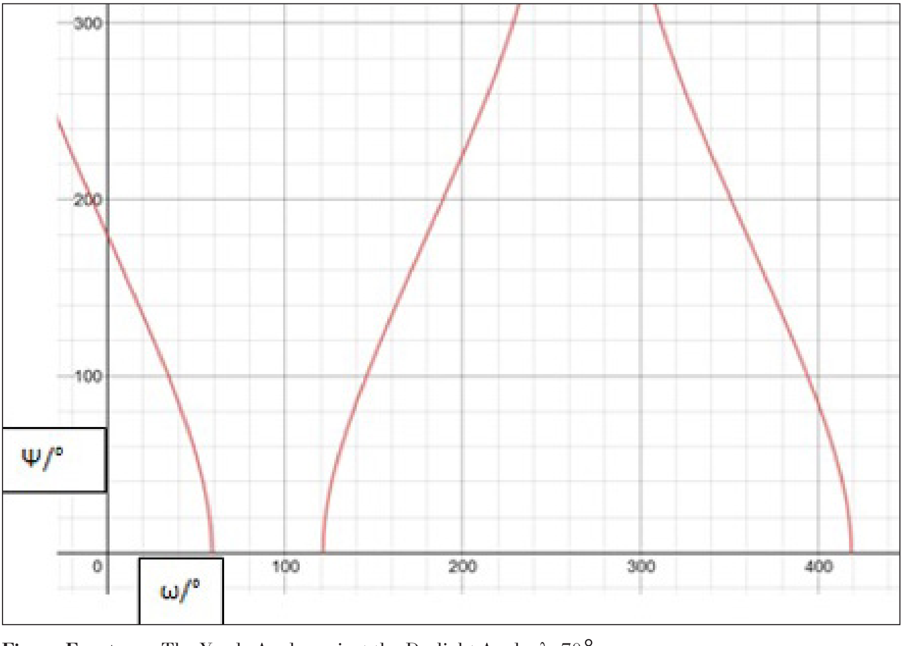


Figure Fourteen : The Yearly Angle against the Daylight Angle. λ=70°

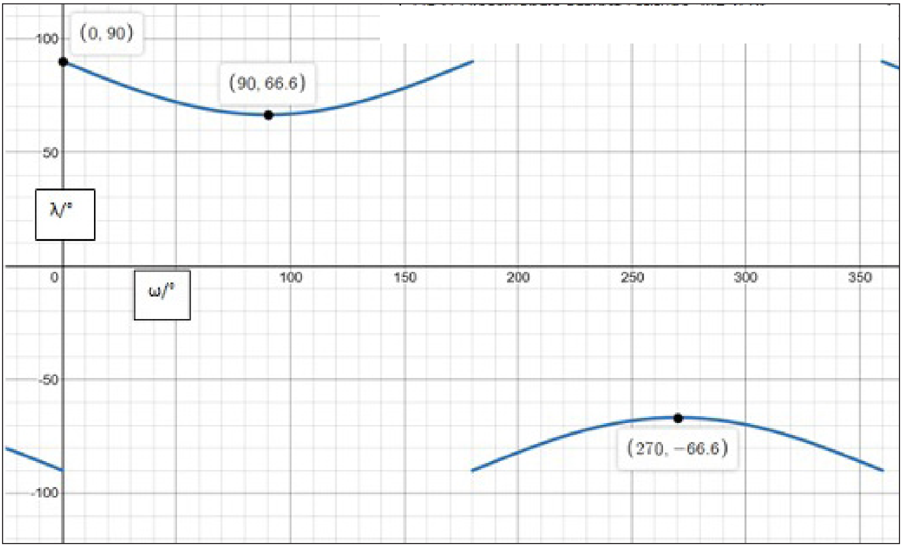


Figure Fifteen : The Yearly Angle against the Daylight Angle. λ=39°

As can be observed in Fig.14, when λ is above the Polar Circles, there is no sunrise or sunset at certain values for ω, so the graph becomes undefined near the solstices. Opposed to Fig.13, as in Fig.14 λ is negative, the Daylight Angle shrinks during the first quarter of the year.

For Fig.15, the lowest latitudes to obey the formula for λ=360°should be equal to the latitudes of the Polar Circles. Our equation for λ predicts a polar circle latitude of ±66.6 to three significant figures, and the published latitudes are ±66°33′47.8″= ±66.563°, so we can conclude the formula makes an accurate prediction.

To try out equation (4c) first-hand, the following measurements were taken and checked against their corresponding GPS coordinates:

Date: 09/11/19	Time of sunrise (GMT): 7:26 = 7.43	Time of sunset (GMT): 17:40 =17.67	Measurement coordinates: (39°18'N, 3°6'E) [15]

			$\boldsymbol{\lambda = 39.3°}$
Ω=days from 21/03/2019 to 09/11/2019. $\Omega = 10 + 3 * 30 + 4 * 31 + 9$ $\Omega = 233$	$\Psi = 15 * (GMT_{sumset} - GMT_{sunrise})$ $\Psi = 15 * (17.67 - 7.4)$ $\Psi = 153.6°$		
$\theta = \arcsin\left(\sin(23.4) * \sin\left(\frac{360}{365.25} * 233\right)\right)$ $\theta = -17.6°$	$\lambda = \arctan\left(-\frac{\cos\left(\frac{153.75}{2}\right)}{\tan(-17.6)}\right)$ $\boldsymbol{\lambda = 35.7°}$		

Within its limitations due to assumptions made, such as constant orbital speed and refraction, and inaccuracies in measurements due to visibility, equation (4c) uses trigonometry to model the relationship between the angle of sunlight, the date and the observer's latitude to a good degree of accuracy, erring by -3.6° (2%).

Calculating Latitude Through the Sunrise or Sunset Bearing

Whilst the Daylight angle method for finding latitude works, it presents several limitations which reduce its usefulness whilst navigating. Firstly, it would take a time period averaging 12 hours to obtain the two measurements required to find your latitude, and until then any course chosen risks travelling in an erroneous direction. Furthermore, to avoid needing any corrections, the observer would have to remain in the same location throughout the measurement, which is unpractical for a travelling vessel, and at sea almost impossible to achieve due to drift, waves, and currents.

Therefore, a different approach was explored building on the Daylight Angle, which uses the bearing of sunrise or sunset to find latitude. As these measurements are instantaneous, it eliminates the difficulties presented by the Daylight Angle formula.

Firstly, compass bearings must be translated onto a spherical perspective. The North-South needle on a compass is

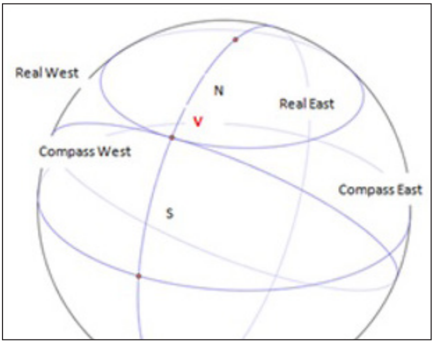


Figure Sixteen : Real vs. compass East and West

tangential to the longitude meridian the observer is on, and the East-West needle is tangential to the observer's parallel. However, to exploit spherical geometry later on, we shall use the bearing of the East-West needle to determine another great circle normal to the longitude great circle, as shown in Fig.16.

For any point V on a sphere, and a body located in its celestial hemisphere, there is a straight line which connects the celestial body to point V. When this celestial body is directly on V's horizon (at 0° altitude), this line is tangential to point V's position on a sphere. As solar rays are assumed to be parallel, this occurs when angle $VO\odot = 90°$.

As can be seen from Fig.18, $V\odot$ (which is tangential to V) is oriented, from V's perspective, on the same bearing as the great circle segment which connects V to M, the point over which the Sun is 90° overhead. This allows us to think about the position of the Sun in terms of M, allowing us to introduce concepts from Spherical Trigonometry to find V's latitude.

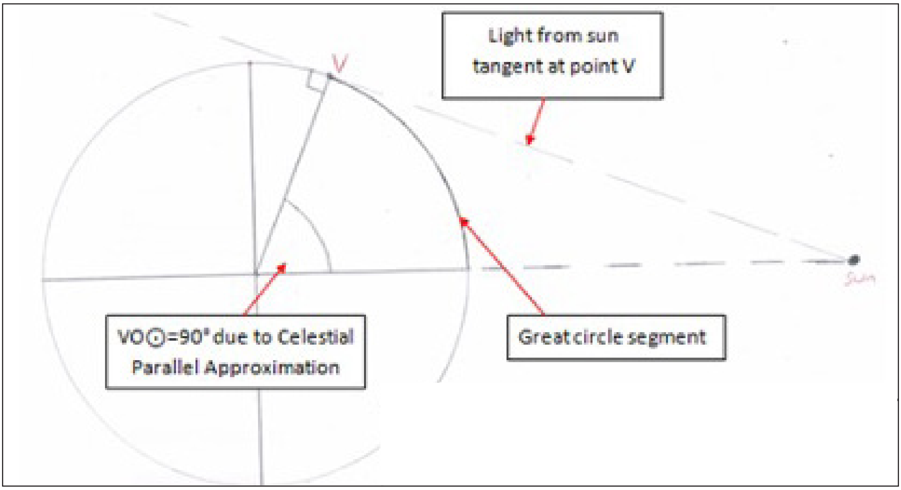


Figure Seventeen : Tangent to V translated into a great circle segment.

We can define a segment VM, which connects V to M. Another segment VN connects V (the observer at sunrise) to N. (When V is at sunset, 360-NVM gives the bearing of M, and therefore the sun). Furthermore, M can be connected to the North Pole using another segment of a great circle, NM. This gives us a spherical triangle, and angle NVM can be found from the existing information.

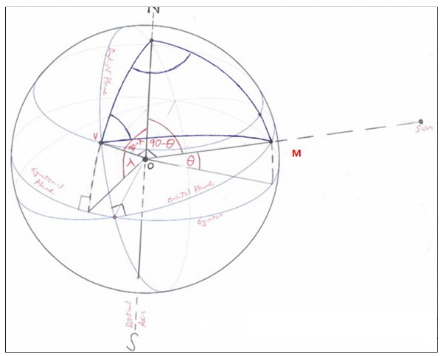


Figure Eighteen : Spherical Triangle NVM

As can be seen from fig.18:

$$VN = 90 - \lambda$$

$$MN = 90 - \theta$$

From the Daylight angle formula, we know Ψ= the angle of daylight received by a point on λV. Therefore, the point on λV which is at midday when V is at sunrise is (Ψ/2)° away from V. As it is midday for a whole meridian at any given time, λ's midday point is on the same meridian as M. NS passes through N, so we can further conclude that:

$$VNM = \frac{\Psi}{2}$$

Moreover, as the Daylight Plane (which V lies on) is normal to ⊙O (which M lies on) we can conclude that:

$$VM = 90°.$$

Proof

We apply the spherical cosine rule: cos(c) = cos(a) * cos(b)+ sin(a) * sin (b) * cos (C) [21]

$$\cos(VM) = \cos(90-\lambda) * \cos(90-\theta) + \sin(90-\lambda) * \sin(90-\theta) * \cos\left(\frac{\Psi}{2}\right)$$

$$\cos(VM) = \sin(\lambda) * \sin(\theta) + \cos(\lambda) * \cos(\theta) * \cos\left(\frac{\Psi}{2}\right)$$

$$\cos(VM) = \sin(\lambda) * \sin(\theta) + \cos(\lambda) * \cos(\theta) * \cos\left(\frac{2\arccos(-\tan(\lambda) * \tan(\theta))}{2}\right)$$

$$\cos(VM) = \sin(\lambda) * \sin(\theta) + \cos(\lambda) * \cos(\theta) * -\tan(\lambda) * \tan(\theta)$$

$$\cos(VM) = \sin(\lambda) * \sin(\theta) - \cos(\lambda) * \cos(\theta) * \frac{\sin(\lambda)}{\cos(\lambda)} * \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cos(VM) = 0, QED.$$

Solving for NVM

First, we rearrange the cosine rule and substitute in the values for MN, VN, and VM quoted above

$$\cos(NVM) = \frac{\cos(MN) - \cos(VN) * \cos(VM)}{\sin(VN) * \sin(VM)}$$

$$\cos(NVM) = \frac{\cos(90-\theta) - \cos(90-\lambda) * \cos(90°)}{\sin(90-\lambda) * \sin(90°)}$$

$$\cos(NVM) = \frac{\sin(\theta) - \sin(\lambda) * 0}{\cos(\lambda) * 1}$$

$$\cos(NVM) = \frac{\sin(\theta)}{\cos(\lambda)}$$

$$\boldsymbol{NVM(sunrise) = \arccos\left(\frac{\sin(\theta)}{\cos(\lambda)}\right)}$$

$$360 - NVM = sunset\ bearing$$

$$\boldsymbol{NVM(sunset) = 360 - \arccos\left(\frac{\sin(\theta)}{\cos(\lambda)}\right)}$$

Rearranging for λ:

$$\cos(NVM) = \frac{\sin\,(\theta)}{\cos\,(\lambda)}$$

$$\frac{\sin\,(\theta)}{\cos\,(NVM)} = \cos\,(\lambda)$$

$$\boldsymbol{\pm\lambda = \arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)}$$

If we substitute the sunset bearing (360°-NVM) for NVM, the expression simplifies to obtain the same formula:

$$\pm\lambda = \arccos\left(\frac{\sin(\theta)}{\cos(360 - NVM)}\right) = \arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)$$

Equation (5c), however, has an important limitation. As can be observed in Fig.19 and Fig.20, for points on two latitudes of equal magnitude, but opposite sign (±λ) the sun will rise and set on the same bearing (as cos(λ) = cos(−λ).) Therefore, this method will limit λ to two locations: one in the Southern Hemisphere and another in the Northern Hemisphere.

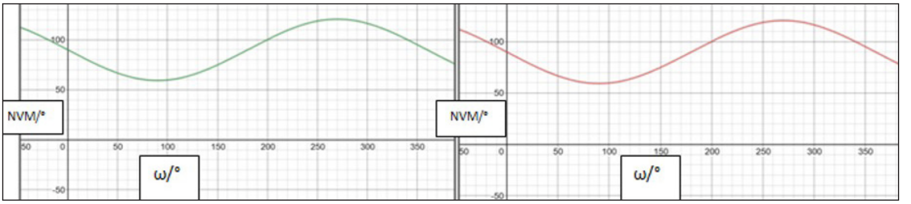


Figure Sixteen : Real vs. compass East and West

Measuring the sun's azimuth some time after sunrise or some time before sunset can help rule out one of the possible latitudes. If the sun travels north (sunrise) or comes from north (sunset), then λ<ϴ.

Conversely, if the sun travels south (sunrise) or comes from south (sunset), then λ>ϴ. Finally, if λ=ϴ, the sun will travel towards the observer's zenith (straight up).

However, this solution does not apply when λ has lesser magnitude than ϴ, because when this is the case, the possible latitudes will either both be above ϴ (when ϴ <0) or both be below ϴ (when ϴ <0). Expressed mathematically, when |λ| < |ϴ|, ±λ > −ϴ or ±λ < +ϴ.

In these cases, additional observations of the sun's bearing do not rule out one of the possible latitudes.

However, the Hemisphere navigators are in is usually well known to them, unless they are navigating close to the equator. In these cases, confirmation through the Solar Noon formula would be required to eliminate one of the possible latitudes, limiting the use of this formula.

First-hand calculations:

Understanding the observer's perspective of sunrise and sunset allows two additional individual measurements to be used to find λ.

The results imply equation (5c) is more accurate than (4c), erring by -3.2° (1.8%). It is speculated this is because atmospheric refraction may have had less impact on bearing measurements.

However, this method requires corroboration through additional measurements under specific circumstances, limiting its use when navigating close to the equator.

NVM (sunrise) =114°	NVM (sunset) =248°	$\theta = -17.6°$
$\lambda = \arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)$ $\lambda = \arccos\left(\frac{\sin(-17.6)}{\cos(114)}\right)$ $\boldsymbol{\lambda = 41.9°}$	$\lambda = \arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)$ $\lambda = \arccos\left(\frac{\sin(-17.6)}{\cos(248)}\right)$ $\boldsymbol{\lambda = 36.1°}$	Measurement location coordinates: (39°18'N, 3°6'E) $\boldsymbol{\lambda = 39.3°}$

Calculating Longitude using Sunrise and Sunset Observations

The times of sunrise and sunset can be used to calculate longitude too, through some manipulation of the formulae for latitude explored above. As 12.0= Solar Noon, it follows that:

$$LST_{sunrise} = 12 - \frac{\frac{\Psi}{2}}{15} \qquad LST_{sunset} = 12.00 + \frac{\frac{\Psi}{2}}{15}$$

Introducing these into the equation φ = 15 * (GMT − LST) gives:

$$\varphi = 15 * \left(GMT_{sunrise} - \left(12 - \frac{\Psi}{30}\right)\right) \qquad \varphi = 15 * \left(GMT_{sunset} - \left(12 + \frac{\Psi}{30}\right)\right)$$

Values for NVM can be used to find λ through the Sunrise/Sunset formula, and this can be used in turn to find Ψ through the Daylight Angle formula. Substituting λ=

$\arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)$ into $\Psi = 2\arccos(-\tan(\theta) * \tan(\lambda))$ gives

$$\Psi = 2\arccos\left(-\tan(\theta) * \tan\left(\arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)\right)\right)$$

Substituting this into $\varphi = 15 * \left(GMT_{sunrise} - \left(12 - \frac{\Psi}{30}\right)\right)$

$$\varphi = 15 * \left(GMT_{sunrise} - \left(12 - \frac{2\arccos\left(-\tan(\theta) * \tan\left(\arccos\left(\frac{\sin(\theta)}{\cos(NVM)}\right)\right)\right)}{30}\right)\right)$$

$$\boldsymbol{\varphi = 15 * GMT_{sunrise} - 180 + \arccos\left(-\tan(\theta) * \tan\left(\arccos\left(\frac{\sin(\theta)}{\cos(NVM_{sunrise})}\right)\right)\right)}$$

Repeating this process for sunset gives :

$$\boldsymbol{\varphi = 15 * GMT_{sunset} - 180 - \arccos\left(-\tan(\theta) * \tan\left(\arccos\left(\frac{\sin(\theta)}{\cos(NVM_{sunset})}\right)\right)\right)}$$

These two formulae allow us to calculate longitude using NVM measurements at sunrise and sunset, along with the time in GMT.

NVM (sunrise) =114°	Time of sunrise (GMT) =7.43	$\theta = -17.6°$	Measurement coordinates: φ=**3.1°** [15]
$\varphi = 15 * 7.43 - 180 + \arccos\left(-\tan(-17.6) * \tan\left(\arccos\left(\frac{sin(-17.6)}{cos(114)}\right)\right)\right)$ $\boldsymbol{\varphi = 4.9°}$			
NVM (sunset) =248°	Time of sunset (GMT) =17.67		
$\varphi = 15 * 17.67 - 180 - \arccos\left(-\tan(-17.6) * \tan\left(\arccos\left(\frac{sin(-17.6)}{cos(248)}\right)\right)\right)$ $\boldsymbol{\varphi = 8.0°}$			

These methods allow us to calculate longitude at three different times in a day, instead of just one, with an error of +1.4% for the worst measurement using (5). However, in absolute terms (+4.9°) this is a more significant error than those obtained for latitude.

Conclusion

Over the years, mathematical models for the solar system have become increasingly accurate, allowing for different ways to calculate an observer's location. This paper has explored three different methods of calculating latitude, using solar height at noon, hours of daylight and the bearing of sunrise and sunset. Furthermore, it has applied these formulas to expand the number of moments an observer can determine their longitude at, to include sunrise and sunset besides solar noon. These methods allow navigators to find their location with minimum equipment (requiring only a compass, sextant, and chronometer set at GMT).

The errors presented for measured values are small enough as to be useful in an emergency, despite not being accurate enough for regular journeys.

Another important limitation is that sunrise and sunset measurements present ambiguity near the equator, but this can be resolved using other methods to corroborate the coordinates obtained.

A possible area for further investigation could be correcting the formulae to account for atmospheric refraction and the earth's varying orbital speed, reducing measurement errors. Moreover, the relationship between the Azimuth and Altitude of the sun (which relies on the Daylight Angle and the Sunrise and Sunset bearings) could be investigated, yielding a formula that would allow an observer to determine latitude and longitude at any time of day, with any solar position.

This model would also correct the Equator ambiguities without relying on multiple measurements. Lastly, other areas of navigation, such as stellar triangulation, and the mathematics of courses and headings on a moving vessel and their conversions to a two-dimensional map, could also be explored. These topics have many applications in other areas, such as maximising natural light in homes and maximising yield in solar farms.
Overall, the methods explored illustrate how mathematical models are highly useful to describe and predict real-life scenarios. Methods
such as these have been a part of human development, travel and exploration throughout the years, and their usefulness in different situations ensures their continued use and study in the future.

BIBLIOGRAPHY

[1] Allen, Carroll W. Astrophysical Quantities. The Athlone Press, 1973.
[2] Anders, William. “Earth Photograph Shot from the Apollo 8, Showing the Daylight Plane.” All of You on the Good Earth, 24 Dec. 1968, eoimages.gsfc.nasa.gov/images/imagerecords/144000/144427/AS08-14-2383_lrg.jpg.
[3] Bowditch, Nathaniel. American Practical Navigator. Defense Mapping Agency Hydrogr./Topogr. Center, 1981.15
[4] Bratcher, Amy J. “Navigation at Sea, History Of.” Water Encyclopedia, 2010, www.waterencyclopedia.com/Mi-Oc/Navigation-at-Sea-History-of.html.
[5] Carlson, T W. “Azimuth-Altitude Schematic.” File:Azimuth-Altitude Schematic.svg, 7 May 2012, commons.wikimedia.org/wiki/File:Azimuth-Altitude_schematic.svg.
[6] Giesen, Juergen. “Transit and Culmination.” Transit and Culmination, 13 Jan. 2017, geoastro.de/TransitCulm/index.html.
[7] Schombert, James. “Earth Coordinate System.” Earth Coordinate System, Mar. 2011, abyss.uoregon.edu/~js/ast121/lectures/lec03.html.
[8] Stern, David P. “Kepler’s Second Law.” NASA, NASA, 7 Apr. 2014, www-istp.gsfc.nasa.gov/stargaze/Skepl2nd.htm.
[9] Stern, David P. “Precession.” NASA, NASA, 10 Oct. 2016, www-istp.gsfc.nasa.gov/stargaze/Sprecess.htm.
[10] Stern, David P. “The Round Earth and Christopher Columbus.” NASA, NASA, 31 Mar. 2014, www-istp.gsfc.nasa.gov/stargaze/Scolumb.htm.
[11] Szabo, Adam. “Latitude and Longitude.” NASA, NASA, 10 Oct. 2016, www-istp.gsfc.nasa.gov/stargaze/Slatlong.htm.
[12] Tanner, Jay. “Obliquity of the Ecliptic, Nutation in Obliquity and Latitudes of the Arctic/Antarctic Circles.” Obliquity of the
Ecliptic and Nutation, 2019, www.neoprogrammics.com/obliquity_of_the_ecliptic/.
[13] Various Contributors. “Earth Is Tilted.” Timeanddate.com, Aug. 2017, www.timeanddate.com/astronomy/axial-tilt-obliquity.html.
[14] Vallado, David A., and Wayne D. McClain. Fundamentals of Astrodynamics and Applications. Microcosm Press, 2013.
[15] Various Contributors. “Google Maps: Cap De Ses Salines.” Google Maps, Google, 2019, www.google.es/maps/place/Cap+de+Ses+Salines/@39.2650354,3.0551532,18.75z/data=!4m5!3m4!1s0x1297008a30488a1f:0x98860f86aef6f576!8m2!3d39.2644274!4d3.0556436?hl=es&authuser=0.
[16] Various Contributors. “RESOLUTION B1&B2.” IAU 2012, 2012, www.iau.org/static/resolutions/IAU2012_English.pdf.
[17] Wallard, Andrew, and Mills Göbel. “The International System of Units.” The International System of Units, Organisation Intergouvernementale De La Convention Du Mètre, 2008, www.bipm.org/utils/common/pdf/si_brochure_8_en.pdf.
[18] Forman, Pearson Scott. “The Longitude Lines on the Globe.” File:Longitude Blue.svg, Wikimedia, 24 Mar. 2011,
[19] Williams, Matt. “What Is the Rotation of the Earth?” Universe Today, 2 Mar. 2019, www.universetoday.

com/47181/earths-rotation/.
[20] Various Contributors. “Stellarium Astronomy Software.” Stellarium Astronomy Software, 2019, stellarium.org/es/.
[21] W. Gellert, S. Gottwald, M. Hellwich, H. Kästner, and H. Küstner, The VNR Concise Encyclopedia of Mathematics, 2nd ed., ch. 12 (Van Nostrand Reinhold: New York, 1989).
Spherical diagrams drawn with Spherical Easel: Austin, David, and William Dickinson. “Spherical Easel.” Download Spherical Easel, 2009, merganser.math.gvsu.edu/easel/download.html.
Graphs drawn with Desmos: Various Contributors. “Explore Math with Desmos.” Desmos.com, 2019, www.desmos.com/.